# C3 Exercise 7C (using double angle trig in proofs)



1a prove that $\dfrac{\cos 2A}{\cos A + \sin A} = \cos A - \sin A$ .

$$\text{LHS} = \frac{\cos 2A}{\cos A + \sin A}$$

$$= \frac{\cos^2 A - \sin^2 A}{\cos A + \sin A}$$

$$= \frac{\cancel{(\cos A + \sin A)}(\cos A - \sin A)}{\cancel{(\cos A + \sin A)}}$$

$$= \cos A - \sin A = \text{RHS}. \quad \square$$

1b prove that $\frac{\sin B}{\sin A} - \frac{\cos B}{\cos A} \equiv 2 \operatorname{cosec} 2A \sin(B-A)$.

$$\text{RHS} = 2 \operatorname{cosec} 2A \sin(B-A)$$

$$= 2\left(\frac{1}{\sin 2A}\right)(\sin B \cos A - \cos B \sin A)$$

$$= \frac{\cancel{2}(\sin B \cos A - \cos B \sin A)}{\cancel{2} \sin A \cos A}$$

$$= \frac{\sin B}{\sin A} - \frac{\cos B}{\cos A} = \text{LHS}. \qquad \square.$$

1c Prove that $\dfrac{1 - \cos 2\theta}{\sin 2\theta} \equiv \tan\theta$.

$$\text{LHS} = \frac{1 - \cos 2\theta}{\sin 2\theta}$$

$$\equiv \frac{1 - (1 - 2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$\equiv \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$\equiv \frac{\cancel{2}\,\cancel{\sin\theta}\,\sin\theta}{\cancel{2}\,\cancel{\sin\theta}\,\cos\theta}$$

$$\equiv \tan\theta. \quad \square$$

1d Prove that $\dfrac{\sec^2\theta}{1-\tan^2\theta} \equiv \sec 2\theta$

$$\text{LHS} \equiv \frac{\sec^2\theta}{1-\tan^2\theta}$$

$$\equiv \frac{1}{\cos^2\theta} \div \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}$$

$$\equiv \frac{1}{\cos^2\theta} \times \frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta}$$

$$\equiv \frac{1}{\cos^2\theta - \sin^2\theta}$$

$$\equiv \frac{1}{\cos 2\theta}$$

$$\equiv \sec 2\theta. \quad \square$$

1e Prove that $2(\sin^3\theta\cos\theta + \cos^3\theta\sin\theta) \equiv \sin 2\theta$

$$\text{LHS} \equiv 2\sin^3\theta\cos\theta + 2\cos^3\theta\sin\theta$$

$$\equiv \sin^2\theta(2\sin\theta\cos\theta) + \cos^2\theta(2\sin\theta\cos\theta)$$

$$\equiv (\sin^2\theta + \cos^2\theta)(2\sin\theta\cos\theta)$$

$$\equiv (1)(\sin 2\theta)$$

$$\equiv \sin 2\theta. \quad \square.$$

1f prove $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$.

$$\text{LHS} = \frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta}$$

$$= \frac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin(3\theta - \theta)}{\sin\theta\cos\theta}$$

$$= \frac{\sin 2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$

$$= 2.$$

notice that the LHS has the general form of $\sin(A-B)$ and exploit this.

1g show

$$\operatorname{cosec}\theta - 2\cot 2\theta\cos\theta \equiv 2\sin\theta$$

Start with $\cot 2\theta = \frac{\cos 2\theta}{\sin 2\theta} = \frac{\cos^2\theta - \sin^2\theta}{2\sin\theta\cos\theta}$

$$\operatorname{cosec}\theta - 2\cot 2\theta\cos\theta \equiv \frac{1}{\sin\theta} - \frac{\cancel{2\cos\theta}\left(\cos^2\theta - \sin^2\theta\right)}{\cancel{2\cos\theta}\,\sin\theta}$$

$$\equiv \frac{1 - \left(1 - \sin^2\theta - \sin^2\theta\right)}{\sin\theta}$$

$$\equiv \frac{\cancel{1 - 1} + 2\sin^2\theta}{\sin\theta} \equiv 2\sin\theta.$$

1h prove $\dfrac{\sec\theta - 1}{\sec\theta + 1} = \tan^2\dfrac{\theta}{2}$.

put $\theta = 2\phi$ then

$$LHS = \frac{\sec 2\phi - 1}{\sec 2\phi + 1} = \frac{\dfrac{1 - \cos 2\phi}{\cos 2\phi}}{\dfrac{1 + \cos 2\phi}{\cos 2\phi}} = \frac{1 - \cos 2\phi}{1 + \cos 2\phi}$$

$$= \frac{1 - (1 - 2\sin^2\phi)}{1 + (2\cos^2\phi - 1)}$$

$$= \frac{2\sin^2\phi}{2\cos^2\phi}$$

$$= \tan^2\phi$$

but $\tan^2\phi = \tan^2\dfrac{\theta}{2} = RHS$. $\square$

1: Prove that $\tan\left(\frac{\pi}{4} - x\right) \equiv \frac{1 - \sin 2x}{\cos 2x}$

$$\text{LHS} \equiv \tan\left(\frac{\pi}{4} - x\right)$$

$$\equiv \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}$$

$$\equiv \frac{1 - \tan x}{1 + \tan x}$$

$$\equiv \left(\frac{\cos x - \sin x}{\cos x}\right) \div \left(\frac{\cos x + \sin x}{\cos x}\right)$$

$$\equiv \frac{\cos x - \sin x}{\cos x + \sin x}$$

$$\equiv \frac{(\cos x - \sin x)(\cos x - \sin x)}{(\cos x + \sin x)(\cos x - \sin x)}$$

$$\equiv \frac{\cos^2 x - 2\sin x\cos x + \sin^2 x}{\cos^2 x - \sin^2 x} \equiv \frac{1 - \sin 2x}{\cos 2x} \equiv \text{RHS}. \square.$$

2a Prove that $\tan\theta + \cot\theta = 2\operatorname{cosec} 2\theta$

$$RHS = 2\operatorname{cosec} 2\theta = \frac{2}{\sin 2\theta}$$

$$= \frac{2}{2\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

THIS IS HARD TO SPOT.

$$= \frac{\cos^{\cancel{2}}\theta}{\sin\theta\cancel{\cos\theta}} + \frac{\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}\cos\theta}$$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \cot\theta + \tan\theta$$

$$= LHS$$

Alternatively ...

2a Prove that $\tan\theta + \cot\theta = 2\operatorname{cosec} 2\theta$

$$\text{LHS} = \tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$

$$= \frac{2}{2\sin\theta\cos\theta}$$

THIS IS HARD TO SPOT.

$$= \frac{2}{\sin 2\theta}$$

$$= 2\operatorname{cosec} 2\theta \quad = \text{RHS}. \ \square$$

But more likely you'll start RHS & meet LHS 'in the middle':

2a Prove that $\tan\theta + \cot\theta = 2\operatorname{cosec} 2\theta$

$$\text{RHS} = 2\operatorname{cosec} 2\theta = \frac{2}{\sin 2\theta}$$

$$= \frac{2}{2\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta\cos\theta}$$ ... get stuck... so start LHS...

$$\text{LHS} = \tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

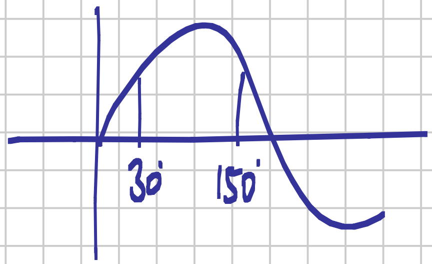
$$= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\sin\theta\cos\theta}$$

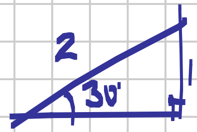
This is ok, provided you set it out this carefully with RHS = LHS = so...

$$\text{RHS} = \frac{1}{\sin\theta\cos\theta} = \text{LHS}$$ so RHS = LHS. □.

2b Hence find the value of $\tan 75° + \cot 75°$

$$\tan 75° + \cot 75° \equiv 2\operatorname{cosec}(2 \times 75°)$$
$$= \frac{2}{\sin 150°}$$
$$\equiv \frac{2}{\sin 30°}$$
$$\equiv \frac{2}{1/2}$$
$$\equiv 4.$$



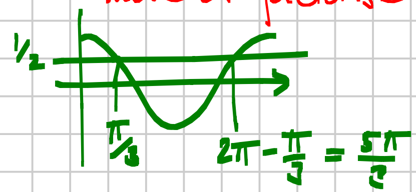



3 Solve the following equations in the interval shown in brackets

a) $\sin 2\theta = \sin\theta$ $\{0 \leq \theta \leq 2\pi\}$

$2\sin\theta\cos\theta = \sin\theta$ use trig identities

$2\sin\theta\cos\theta - \sin\theta = 0$ must not cancel: move & factorise



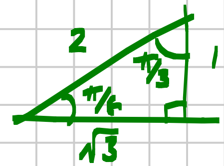

$\sin\theta(2\cos\theta - 1) = 0$

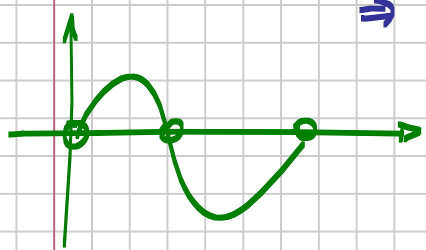
$\Rightarrow$ either $\sin\theta = 0$ or $2\cos\theta - 1 = 0$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = 0, \pi$ or $2\pi$

$\Rightarrow \theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$



$\Rightarrow \theta = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi.$

3b $\cos 2\theta = 1 - \cos\theta \qquad \{-180^\circ < \theta \leq 180^\circ\}$

$\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$ or $\equiv 2\cos^2\theta - 1$ or $\equiv 1 - 2\sin^2\theta$

which is best?

$\cos^2\theta - \sin^2\theta = 1 - \cos\theta$

$2\cos^2\theta - 1 = 1 - \cos\theta$ ✓ This one … then it's all in 'cos'

$1 - 2\sin^2\theta = 1 - \cos\theta$

$\Rightarrow \quad 2\cos^2\theta - 1 = 1 - \cos\theta$

$2\cos^2\theta + \cos\theta - 2 = 0$

it may help now to think 'c' $= \cos\theta$

$2c^2 + c - 2 = 0$

$\Rightarrow \quad c = \dfrac{-1 \pm \sqrt{1^2 - 4(2)(-2)}}{2(2)} = \dfrac{-1 \pm \sqrt{17}}{4}$

3b (ctd)

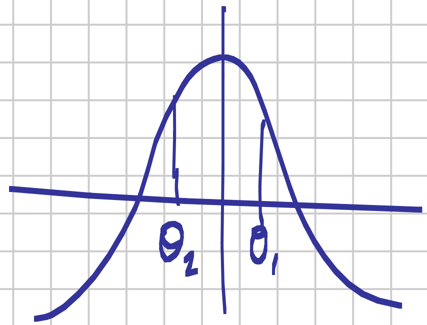
now $\frac{-1-\sqrt{17}}{4} < -1$ so cannot be a solution to $\cos\theta$

but

$$\cos\theta = \frac{-1+\sqrt{17}}{4}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{-1+\sqrt{17}}{4}\right) = 38.7^\circ$$

$$\theta_2 = -\theta_1 = -38.7^\circ$$



3c Solve $3\cos 2\theta = 2\cos^2\theta$ in the interval $0° \leq \theta < 360°$

$$3\cos 2\theta = 2\cos^2\theta$$

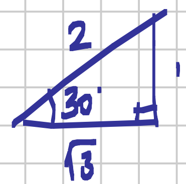
$$3(2\cos^2\theta - 1) = 2\cos^2\theta$$

$$6\cos^2\theta - 3 = 2\cos^2\theta$$

$$4\cos^2\theta - 3 = 0$$

$$\cos^2\theta = \frac{3}{4}$$

$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$

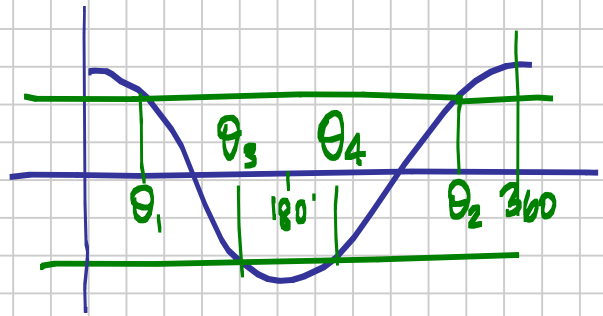



$\theta_1 = 30°$

$\theta_2 = 330°$

$\theta_3 = 150°$

$\theta_4 = 210°$



3d Solve $\sin 4\theta = \cos 2\theta$ for $0 \leq \theta \leq \pi$

$\Rightarrow \quad 0 \leq 2\theta \leq 2\pi$

$\sin 4\theta = \cos 2\theta$

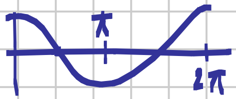
$\Rightarrow \quad 2\sin 2\theta \cos 2\theta = \cos 2\theta$

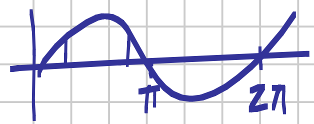
$\Rightarrow \quad 2\sin 2\theta \cos 2\theta - \cos 2\theta = 0$

$\Rightarrow \quad \cos 2\theta (2\sin 2\theta - 1) = 0$

$\Rightarrow \quad \cos 2\theta = 0$ or $\sin 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ or $2\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$




$\Rightarrow \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

$\Rightarrow \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$

so $\theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}$ or $\frac{3\pi}{4}$.

3e Solve $2\tan 2y \tan y = 3$ for $0 \le y < 360°$

$$2\tan 2y \tan y = 3$$

$$2\left(\frac{2\tan y}{1-\tan^2 y}\right)\tan y = 3$$

$$4\tan^2 y = 3 - 3\tan^2 y$$

$$7\tan^2 y = 3$$

$$\tan^2 y = \frac{3}{7}$$

$y = \tan^{-1}\left(\sqrt{\frac{3}{7}}\right)$ or $y = \tan^{-1}\left(-\sqrt{\frac{3}{7}}\right)$

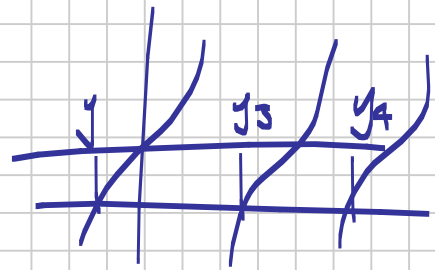

$y_1 = 33.2°$

$y_2 = 213.2°$

$y = -33.2°$

$y_3 = 146.8°$

$y_4 = 326.8°$



3f Solve $3\cos\theta - \sin\frac{\theta}{2} - 1 = 0$ for $0 \leq \theta < 720°$

put $\theta = 2\phi$ then

$3\cos 2\phi - \sin\phi - 1 = 0$ for $0 \leq \phi < 360$

$3(1 - 2\sin^2\phi) - \sin\phi - 1 = 0$

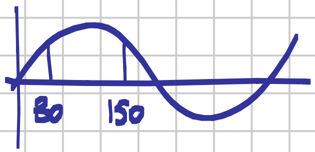
$3 - 6\sin^2\phi - \sin\phi - 1 = 0$

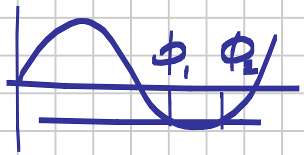
$\Rightarrow \quad 6\sin^2\phi + \sin\phi - 2 = 0$

$\Rightarrow \quad (2\sin\phi - 1)(3\sin\phi + 2) = 0$

$\Rightarrow \quad \sin\phi = \frac{1}{2}$ or $\sin\phi = -\frac{2}{3}$

$\Rightarrow \quad \phi = 30°$ or $150°$ or $\phi_1 = 221.8°$ or $\phi_2 = 318.2°$



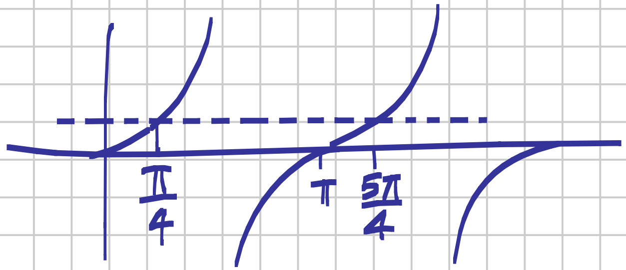

$\Rightarrow \quad \theta = 60°$ or $300°$ $\quad \theta = 443.6°$ or $632.4°$

3g $\cos^2\theta - \sin 2\theta = \sin^2\theta \qquad 0 < \theta \leq \pi$

$$\cos^2\theta - \sin^2\theta - \sin 2\theta = 0$$

$$\cos 2\theta - \sin 2\theta = 0$$

$$\Rightarrow \tan 2\theta = 1 \qquad \text{for } 0 \leq 2\theta \leq 2\pi$$




$$2\theta = \frac{\pi}{4} \quad \text{or} \quad \frac{5\pi}{4}$$

So $\theta = \frac{\pi}{8}$ or $\frac{5\pi}{8}$

3i Solve $2\sin 2\theta = 3\tan\theta$